Solution of Previous Paper of GATE

GATE-2017

1 Indentical charges q are placed at five vertices of a regular hexagon of side a. The magnitude of the electric field and the electrostatic potential at the centre of the hexagon are respectively

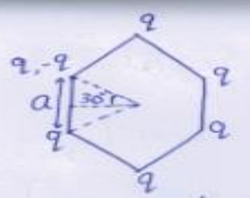
(a) 0, 0

(b) $\frac{q}{4\pi\varepsilon_0 a^2}, \frac{q}{4\pi\varepsilon_0 a}$

(c) $\frac{q}{4\pi\varepsilon_0 a^2}, \frac{5q}{4\pi\varepsilon_0 a}$

(d) $\frac{\sqrt{5}q}{4\pi\varepsilon_0 a^2}, \frac{\sqrt{5}q}{4\pi\varepsilon_0 a}$

sol^n:-



$\because \sin\theta = \frac{L}{K}$

$K = \frac{L}{\sin 30}$

$= \frac{a}{2\sin 30} = a$

$\Rightarrow \vec{E}$ due to $-q$ charge

$\vec{E} = \frac{K(-q)}{a^2}$

$|\vec{E}| = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}$

and other 6(+q) charges cancel out $\vec{E}$ due to each other due to same magnitude but opposite dir^n. So, due to 6(+q) charges resultant $\vec{E}$ is zero.

Potential is scalar quantity. Then potential due to 5(+q) charges

$V = \frac{K(5q)}{a}$

$V = \frac{5q}{4\pi\varepsilon_0 a}$

option. (c).

2 A parallel place capacitor with square plates of side 1 m separated by 1 micro meter is filled with a medium of dielectric constant of 10. If the changes on two plates are 1 C and –1C, the voltage across the capacitor is ……………… kV. (upto two decimal places) ($\varepsilon_0 = 8.854 \times 10^{-12} F/m$)

sol^n:-

$$C = \frac{Q}{V} = \frac{\varepsilon A}{d}$$

$$V = \frac{Qd}{\varepsilon A} = \frac{Qd}{\varepsilon_0 \varepsilon_r A} = \frac{1 \times 10^{-6}}{\varepsilon_0 \times 10 \times (1 \times 1)}$$

$$= \frac{10^{-7}}{8.85 \times 10^{-12}} = 1.129 \times 10^4 V$$

$$= 11.29 \text{ kV}$$

3 Light is incident from a medium of refractive index $n = 1.5$ onto vaccum. The smallest angle of incidence for which the light is not transmitted into vaccum is ……………… degrees. (upto two decimal places).

sol^n:- $n = 1.5$

Then angle for which light is completely reflected

critical angle $\theta = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

$= \sin^{-1}\left(\frac{1}{1.5}\right)$

$\theta = 41.81$

4 A monochromatic plane wave in free space with electric field amplitude of 1 V/m is normally incident of a fully reflecting mirror. The pressure exerted on the mirror is ……………… $\times 10^{-12}$ Pa. (up to two decimal places) $\varepsilon_0 = 8.854 \times 10^{-12} F/m$).

sol^n:-

$E = 1$ V/m

∵ for fully reflecting

$$P = \frac{2I}{c}$$

$$= \frac{2}{c} c \langle u \rangle = 2\langle u \rangle$$

$$P = 2 \times \frac{1}{2} \varepsilon_0 E_0^2$$

$$= \varepsilon_0 E_0^2$$

$$= 8.85 \times 10^{-12} \times (1)^2$$

$$P = 8.85 \times 10^{-12} \text{ Pa}$$

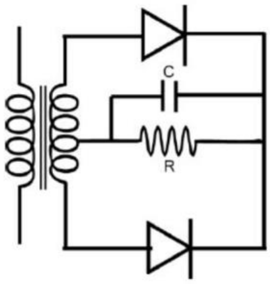
5 The best resolution that a 7 bit A/D converter with 5 V full scale can achieve is ……………. mV. (up to two decimal places).

$$\text{Resolution} = \frac{\text{full voltage scale}}{2^n}$$

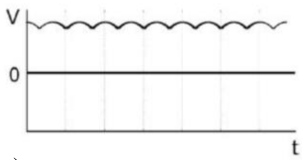
$$= \frac{5}{2^7} \quad [\because n \to \text{no. of bit}]$$

$$= 39.06 \text{ mV}$$

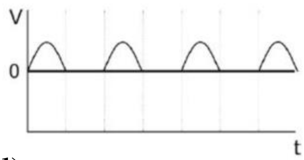
6 In the figure given below, the input to the primary of the transformer is a voltage varying sinusoidally with time. The resistor R is connected to the centre tap of the secondary. Which one of the following plots represents the voltage across the resistor R as a function of time?



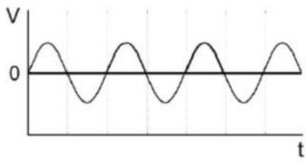
(a)



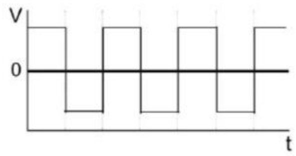
(b)



(c)



(d)



ANS-(A)

7. The atomic mass and mass density of Sodium are 23 and 0.968 g cm^{-3}, respectively. The number density of valence electron is ………… × 10^{22} cm^3. (Up to two decimal places). (Avagadro number, $N_A = 6.022 \times 10^{23}$)

$$\text{Number density } n = \frac{\rho N_A}{M_A}$$

$$= \frac{0.968 \times 6.022 \times 10^{23}}{23}$$

$$n = 2.53 \times 10^{22}$$

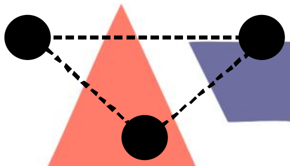
8. Consider a one-dimensional lattice with a weak periodic potential $U(x) = U_0 \cos\left(\frac{2\pi x}{a}\right)$. The gap at the edge of the Brillouin zone $\left(k = \frac{\pi}{a}\right)$ is ;

(a) U_0 (b) $\frac{U_0}{2}$

(c) $2U_0$ (d) $\frac{U_0}{4}$

ANS-(A)

9. Consider a triatomic molecule of the shape shown in the figure below in three dimensions. The heat capacity of this molecule at high temperature much higher than the vibrational and rotational energy scales of the molecule but lower than its bond dissociation energies) is :



(a) $\frac{3}{2}k_B$ (b) $3k_B$

(c) $\frac{9}{2}k_B$ (d) $6k_B$

Normal temp. d.o.f. of non-linear triatomic molecule

$f = 3N - k$

$= 3\times 3 - 3$

$f = 6$

So, energy $E = 6 \times \frac{1}{2} K_B T$

$E = 3K_B T$

But, at high temp. 3 extra vibrational d.o.f.

So, vibrational energy $= 3K_B T$

Total energy $U = 3K_B T + 3K_B T$

$U = 6K_B T$

Heat capacity $C_V = \left(\frac{\partial U}{\partial T}\right)_V$

$C_V = 6K_B$

option (d)

10. If the Lagrangian $L_0 = \frac{1}{2}m\left(\frac{dq}{dt}\right)^2 - \frac{1}{2}m\omega^2 q^2$ is modified to $L = L_0 + \alpha q\left(\frac{dq}{dt}\right)$, which one of the following is TRUE ?

(a) Both the canonical momentum and equation of motion do not change

(b) Canonical momentum changes, equation of motion does not change

(c) Canonical momentum does not change, equation of motion changes

(d) Both the canonical momentum and equation of motion change

sol^n:-

$$L_0 = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \quad \text{--- } (1)$$

$$\therefore L = L_0 + \alpha q\dot{q} \quad \text{--- } (2)$$

$$P = \frac{\partial L}{\partial \dot{q}} = m\dot{q} + \alpha q \quad \text{--- } (3)$$

$$P_0 = \frac{\partial L_0}{\partial \dot{q}} = m\dot{q} \quad \text{--- } (4)$$

then Lagrangian eq^n of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0$$

$$\frac{d}{dt}\left(m\dot{q} + \alpha q\right) - \left(-m\omega^2 q + \alpha \dot{q}\right) = 0$$

$$\ddot{q} + \omega^2 q = 0$$

Lagrangian eq^n of motion does not change but from eq^n (3) canonical momentum change.

option (b).

11. Two identical masses of 10 gm each are connected by a massless spring of spring constant 1 N/m. The non-zero angular eigenfrequency of the system is …………………….. rad/s. (up to two decimal places).

$k = 1\,N/m$ $\qquad m = 10\,gm$

$\omega = \sqrt{\frac{k}{m}}$

$m = \frac{m_1 m_2}{m_1 + m_2}$

$= \frac{10 \times 10}{20}$

$m = 5\,gm = 0.005\,kg$

$\omega = \sqrt{\frac{1}{0.005}} = \sqrt{\frac{10^3}{5}}$

$\omega = \sqrt{2} \times 10 = 14.14$ rad/sec.

12. The phase space trajectory of an otherwise free particle bouncing between two hard walls elastically in one dimension is a

(a) straight line (b) parabola

(c) rectangle (d) circle

ANS-(C)

13. The Poisson bracket $[x, xp_y + yp_x]$ is equal to :

(a) $-x$ (b) y

(c) $2p_x$ (d) p_y

$[x, xP_y + yP_x]$

$= [x, xP_y] + [x, yP_x]$

$= 0 + y[x, P_x]$

$= y$

14. The wavefunction of which orbital is spherically symmetric :

(a) p_x (b) p_y

(c) S (d) d_{xy}

ANS-(C)

15. The contour integral $\oint \frac{dz}{1+z^2}$ evaluated along a contour going from $-\infty$ to $+\infty$ along the real axis and closed in the lower half-plane by a half circle is equal to ………………… . (up to two decimal places).

sol^n:-

Since here is not affected in the lower half plane because

$\oint \frac{dz}{1+z^2}$ $f(z) = \frac{1}{1+z^2}$

poles $z^2+1=0$

$z^2=-1$

$z=\pm i$

for lower half plane

$z=-i$

$$\text{Residue }(R) = \lim_{z\to -i}(z+i)f(z)$$

$$= \lim_{z\to -i}\frac{1}{z-i}$$

$$= \frac{-1}{2i}$$

$$\oint f(z)\,dz = -2\pi i(\Sigma R_i)$$

$$= -2\pi i\left(-\frac{1}{2i}\right)$$

$$= \pi$$

$$\oint f(z)\,dz = 3.14$$

16. The Compton wavelength of a portion is ……………..fm.(up to two decimal places). ($m_p = 1.67 \times 10^{-27}$ kg, h - 6.626×10^{-34}Js, $e = 1.602 \times 10^{-19}$C. $c = 3 \times 10^8$ ms^{-1})

$$\lambda = \frac{h}{m_o c}$$

$$= \frac{6.67\times 10^{-34}}{1.67\times 10^{-27}\times 3\times 10^8}$$

$$= 1.32\times 10^{-15}\,m$$

$$\lambda = 1.32\,fm$$

17. Which one of the following conservation laws is violated in the decay $\tau^{+} \rightarrow \mu^{+}\mu^{+}\mu^{-}$

(a) Angular momentum (b) Total Lepton number

(c) Electric charge (d) Tau number

sol^n:-

$\tau^{+} \longrightarrow \mu^{+}\mu^{+}\mu^{-}$

L: -1 -1 -1 $+1$ conserved

L_τ: -1 0 0 0 not conserved

L_μ: 0 -1 -1 $+1$ not conserved

Tau Number

18. Electromagnetic interactions are :

(a) C conserving

(b) C non-conserving but CP conserving

(c) CP non-conserving but CPT conserving

(d) CPT non-conserving

ANS-(A)

19. A one dimensional simple harmonic oscillator with Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2$ is subjected to a small perturbation, $H_1 = \alpha x + \beta x^3 + \gamma x^4$. The first order correction to the ground state energy is dependent on:

(a) only β (b) α and γ

(c) α and β (d) only γ

^{n}:-

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$H' = \alpha x + \beta x^3 + \gamma x^4$$

$$E^{(1)} = \langle\psi|H'|\psi\rangle = \int \psi^* H' \psi \, dx$$

$\because$ due to αx and βx^3 term $E^{(1)} = 0$ (anti-symmetric)

then $E^{(1)}$ is depend only on γ.

20. For the Hamiltonian $H = a_0 I + \vec{b}.\vec{\sigma}$ where $a_0 \in R$, $\vec{b}$ is a real vector, I is the 2×2 identity matrix, and $\vec{\sigma}$ are the Pauli matrices,t he ground state energy is:

(a) | b | (b) $2a_0 - |b|$

(c) $a_0 - |b|$ (d) a_0

sol:-

$$H = a_0 I + \vec{b}\cdot\vec{\sigma}$$

$$= a_0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}b_z & b_x - ib_y\\ b_x + ib_y & -b_z\end{pmatrix}$$

$$H = \begin{pmatrix}a_0 + b_z & b_x - ib_y\\ b_x + ib_y & a_0 - b_z\end{pmatrix}$$

$$|H - \lambda I| = \begin{vmatrix}a_0 + b_z - \lambda & b_x - ib_y\\ b_x + ib_y & a_0 - b_z - \lambda\end{vmatrix} = 0$$

$$\Rightarrow (a_0 + b_z - \lambda)(a_0 - b_z - \lambda) - (b_x^2 + b_y^2) = 0$$

$$\Rightarrow a_0^2 - b_z^2 - 2a_0\lambda + \lambda^2 - (b_x^2 + b_y^2) = 0$$

$$\Rightarrow \lambda^2 - 2a_0\lambda + a_0^2 - b^2 = 0$$

$$\lambda = \frac{2a_0 \pm \sqrt{4a_0^2 - 4(a_0^2 - b^2)}}{2}$$

$$\lambda = a_0 \pm |b| \qquad \left[\because |b| = \sqrt{b_x^2 + b_y^2 + b_z^2}\right]$$

Then, there is ground state energy

$$\lambda = a_0 - |b|$$

21. The coefficient of e^{ikx} in the Fourier expansion of $u(x) = A\sin^2(ax)$ for $k = -2\alpha$ is

(a) A/4 (b) –A/4

(c) A/2 (d) – A/2

Sol^n:-

Coefficient of e^{ikx}

$$I_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{ikx}\,dx$$

$$f(x) = A\sin^2(\alpha x)$$

$$\sin\alpha x = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}$$

$$f(x) = A\sin^2(\alpha x) = A\left(\frac{e^{i\alpha x} - e^{-i\alpha x}}{2i}\right)^2$$

$$= \frac{A}{-4}\left(e^{2i\alpha x} + e^{-2i\alpha x} - 2\right)$$

$$I_k = \frac{1}{2\pi}\left(\frac{-A}{4}\right)\int_{-\pi}^{\pi}\left(e^{2i\alpha x} + e^{-2i\alpha x} - 2\right) e^{ikx}\,dx$$

at $k = -2\alpha$

$$I_k = \frac{-A}{8\pi}\int_{-\pi}^{\pi}\left(e^{-ikx} + e^{ikx} - 2\right) e^{ikx}\,dx$$

$$= \frac{-A}{8\pi}\int_{-\pi}^{\pi}\left(1 + e^{2ikx} - 2e^{ikx}\right) dx$$

$$= \frac{-A}{8\pi}\left\{[x]_{-\pi}^{\pi} + 0 + 0\right]$$

$$I_k = -\frac{A}{4}$$

22. The degeneracy of the third energy level of a 3-deimensional isotropic quantum harmonic oscillator is :

(a) 6 (b) 12

(c) 8 (d) 10

$$= \frac{(n+1)(n+2)}{2} \qquad [\because n = 2]$$

$$\Rightarrow \frac{3\times 4}{2} = 6$$

23. The electronic ground state energy of the Hydrogen atom is –13.6 eV. The highest possible electronic enery eigenstate has an energy equal to :

(a) 0 (b) 1 eV

(c) + 13.6 eV (d) ∞

24. A reversible Carnot engine is operated between temperatures T_1 and T_2 ($T_2 > T_1$) with a photon gas as the working substance. the efficiency of the engine is :

(a) $1-\frac{3T_1}{4T_2}$ (b) $1-\frac{T_1}{T_2}$

(c) $1-\left(\frac{T_1}{T_2}\right)^{3/4}$ (d) $1-\left(\frac{T_1}{T_2}\right)^{4/3}$

ANS-(B)

25. In the nuclear reaction $^{13}C_6 + \nu_e \rightarrow {}^{13}N_7 + X$, the particle X is

(a) an electron (b) an anti-electron

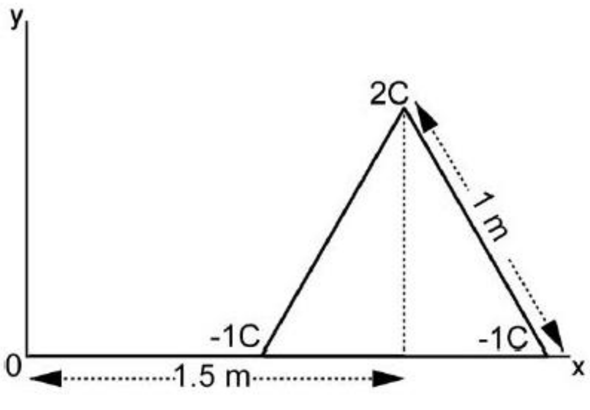
(c) a muon (d) a pion

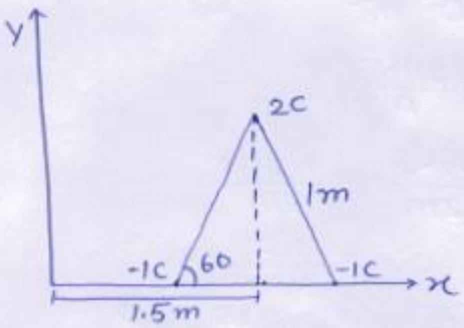
$^{13}C_6 + \nu_e \longrightarrow {}^{13}N_7 + X$

here $X \rightarrow e^-$

for electron lepton no. conservation

26. Three charges (2C, −1 C, −1 C) are placed at the vertices of an equilateral triangle of side 1m as shown in the figure. The component of the electric dipole moment about the marked origin along the $\hat{y}$ direction is ……………… C m.





$\Sigma Q = 0$

So, dipole moment is independent of choice of origin

$P = \Sigma q_i d_i$

$= (-1)\cdot 0 + (-1)(1)\hat{x} + 2(1)[\cos 60\,\hat{x} + \sin 60\,\hat{y}]$

$P_y = 2\sin 60\,\hat{y}$

$P_y = 2 \times \frac{\sqrt{3}}{2}\,\hat{y} = \sqrt{3}\,\hat{y}$

$P = 1.732$

27. An infinite solenoid carries a time varying current I(t) = At^2, with $A \neq 0$. The axis of the solenoid is along the $\hat{z}$ direction. $\hat{r}$ and $\hat{\theta}$ are the usual radial and polar direction in cylindrical polar coordinates. $\vec{B} = B_r\hat{r} + B_\theta\hat{\theta} + B_z\hat{z}$ is the magnetic field at a point outside the solenoid. Which one of the following statements it true?

(a) B_r - 0, $B_\theta = 0, B_z = 0$
(b) $B_r \neq 0, B_\theta \neq 0, B_z = 0$
(c) $B_r \neq 0, B_\theta \neq 0, B_z \neq 0$
(d) $B_r = 0, B_\theta = 0, B_z \neq 0$

ANS-(D)

28. A uniform volume charge density is placed inside a conductor (with resistivity $10^{-2}\Omega m$). The charge density becomes 1/(2.718) of its original value after time .... femto seconds. (up to two decimal places) $\varepsilon_0 = 8.854 \times 10^{-12} F/m)$

$\rho = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}$

then $t = \frac{\varepsilon}{\sigma} = \varepsilon\rho$

$= 8.854 \times 10^{-12} \times 10^{-2}$

$= 88.54 \times 10^{-15}$

$= 88.54$ fm sec.

29. Water freezes at 0°C at atmospheric pressure (1.01×10^5 Pa). The densitites of water and ice at this temperature and pressure are 1000 kg/m^3 and 934 kg/m^3 respectively. The latent heat

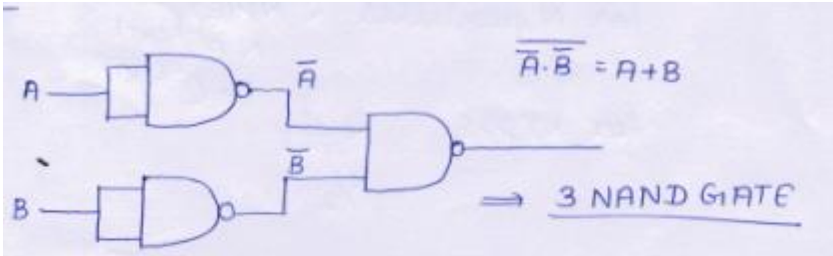
of fusion is 3.34×10^5 J/kg. The pressure required for depressing the melting temperature of ice by 10°C is ……….. GPa.(up to two decimal places)

ANS-(0.15 to 0.19) Using clausius clapeyron equation

30. The minimum number of NAND gates required to construct an OR gate Is :

(a) 2 (b) 4

(c) 5 (d) 3



31. Consider a 2-dimensional electron gas with a density of 10^{19} m^{-2}. The Fermi energy of the system is …………… eV (up to two decimal places).

$$(m_e = 9.31\times10^{-31} kg,\ h = 6.626\times10^{-34} Js,\ e = 1.602\times10^{-19} C)$$

$$\sigma = 10^{19}\ m^{-2}$$

$$k_f = \left(\frac{2\pi N}{A}\right)^{1/2}$$

$$E = \frac{\hbar^2 k_f^2}{2m} = \frac{\hbar^2 2\pi\sigma}{2m}$$

$$= \frac{h^2\sigma}{4\pi m}$$

$$E = \frac{(6.26)^2\times10^{-68}\times10^{19}}{4\times3.14\times9.31\times10^{-31}\times1.6\times10^{-19}}$$

$$\boxed{E_f = 2.09\ eV}$$

32. The total energy of an inert-gas crystal is given by $E(R) = \frac{0.5}{R^{12}} - \frac{1}{R^6}(\text{in } eV)$, where R is the inter-atomic spacing in Angstroms. The equilibrium separation between the atoms is …………………. Angstroms. (up to two decimal places)

$$E(R) = \frac{0.5}{R^{12}} - \frac{1}{R^6}$$

$$\frac{\partial E}{\partial R} = \frac{-12(0.5)}{R^{13}} + \frac{6}{R^7} = 0$$

$$\Rightarrow \boxed{R = 1\text{Å}}$$

33. Consider N non-interacting, distinguishable particles in a two-level system at temperature T. The energies of the levels are 0 and ε, where $\varepsilon > 0$. In the high temperature limit $(k_B T >> \varepsilon)$, what is the population of particles in the level with energy ε ?

(a) $\frac{N}{2}$

(b) N

(c) $\frac{N}{4}$

(d) $\frac{3N}{4}$

Probability of a particle in ε energy level

$$P(\varepsilon) = \frac{e^{(-\varepsilon/K_B T)}}{(1+e^{-\varepsilon/KT})}$$

for N particles $= NP(\varepsilon)$

$$= \frac{N e^{(-\varepsilon/KT)}}{1+e^{(-\varepsilon/KT)}}$$

for $KT >> \varepsilon$ $= N\frac{1}{1+1}$

$$= N/2$$

34. A free electron of energy 1 eV is incident upon a one-dimensional finite potential step of height 0.75 eV. The probability of its reflection from the barrier is .... (up to two decimal places).

$E = 1eV$

$V_0 = 0.75V$

$E > V_0$, then

$$R = \left|\frac{k-k'}{k+k'}\right|^2$$

$$= \left|\frac{1-k'/k}{1+k'/k}\right|^2 \qquad \left[\because\ k = \sqrt{\frac{2mE}{\hbar^2}}\right.$$

$$= \left|\frac{1-\sqrt{1-V_0/E}}{1+\sqrt{1-V_0/E}}\right| \qquad \left.k' = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}\right.$$

$$R = \left|\frac{1-0.5}{1+0.5}\right|^2 = \frac{1}{9} = 0.11$$

35. Consider a one-dimensional potential well of width 3nm. Unsing the uncertainty principle $\left(\Delta x \cdot \Delta p \geq \hbar/2\right)$, an estimate of the minimum depth of the well such that it has at least one bound state for an electron is $(m_e = 9.31 \times 10^{-31} kg,\ h = 6.626 \times 10^{-34} J\,s,\ e = 1.602 \times 10^{-19} C)$;

(a) $1\mu eV$ (b) 1 meV

(c) 1 eV (d) 1 MeV

Bound State

$V_0 > E$

$$E = \frac{p^2}{2m}$$

$$\Delta p \simeq \frac{\hbar}{2\Delta x} \simeq \frac{\hbar}{2a}$$

$$E_{min} = \frac{\Delta p^2}{2m} = \left(\frac{\hbar}{2a}\right)^2 \frac{1}{2m} = \frac{\hbar^2}{8ma^2} = \frac{(1.05\times10^{-34})^2}{8\times9.1\times10^{-31}\times9\times10^{-18}}$$

$$E = 1.06\, meV$$

$$\boxed{V_0 = 1 meV}$$

36. Consider a metal with free electron density of 6×10^{22} cm^{-3}. The lowest frequency electromagnetic radiation to which this metal is transparent is 1.38×10^{16} Hz. If this metal had a free electron density of 1.8×10^{23} cm^{-3} instead, the lowest frequency electromagnetic radiation to which it would be transparent is .... $\times 10^{16}$ Hz. (up to two decimal places).

Cut-off frequency $\propto n^{1/2}$

where $n \rightarrow$ no. density

$$\frac{\omega_1}{\omega_2} = \left(\frac{n_1}{n_2}\right)^{1/2}$$

$$\frac{1.38\times10^{6}}{\omega_2} = \left(\frac{6\times10^{22}}{1.8\times10^{23}}\right)^{1/2}$$

$$\omega_2 = 1.38\times\sqrt{3}\times10^{16} = 2.39\times10^{16}\ \text{Hz}$$

37. An object travels along the x-direction with velocity $c/2$ in a frame O. An observer in a frame O' sees the same object travelling with velocity $c/4$. The relative velocity of O' with respect to O in units of c is .... (up to two decimal places).

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u_x = \frac{c}{2}, \quad u'_x = \frac{c}{4} \qquad v = ?$$

$$v = \frac{u_x - u'_x}{1 - \frac{u_x u'_x}{c^2}}$$

$$= \frac{\frac{c}{2} - \frac{c}{4}}{1 - \frac{c^2}{8c^2}} = \frac{c}{4} \Big/ 1 - \frac{1}{8}$$

$$v = \frac{2c}{7} = 0.28c$$

38. The integral $\int_0^\infty x^2 e^{-x^2} dx$ is equal to .... (up to two decimal places).

$$\int_0^\infty x^2 e^{-x^2} dx = ?$$

$$\int_0^\infty x^m e^{-\alpha x^n} dx = \frac{1}{n} \frac{\Gamma\frac{m+1}{n}}{(\alpha)^{\frac{m+1}{n}}}$$

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{1}{2} \frac{\Gamma\frac{3}{2}}{(1)^{3/2}} = \frac{1}{2} \times \frac{1}{2}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{4} = 0.443$$

39. The imaginary part of an analytic complex function is $v(x, y) = 2xy + 3y$. The real part of the function is zero at the origin. The value of the real part of the function at $1 + i$ *is* .... (up to two decimal places).

$$v(x, y) = 2xy + 3y$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + 3$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$du = (2x+3)dx - 2y\,dy$$

$$u = \int (2x+3)dx - 2\int y\,dy$$

$$= \frac{2x^2}{2} + 3x - 2\frac{y^2}{2}$$

$$u = x^2 + 3x - y^2$$

$$u|_{1+i} = 1 + 3 - 1$$

$$u = 3$$

40. Let X be a column vector of dimension $n > 1$ with at least one non-zero entry. The number of non-zero eigenvalues of the matrix $M = XY^T$ *is :*

(a) 0 (b) n

(c) 1 (d) $n - 1$

sol:- X - column vector with at least one non-zero entry.
Using principle of mathematical induction for $n = 2$ $(n > 1)$

$$X = \begin{bmatrix} a \\ 0 \end{bmatrix} \qquad X^T = [a \quad 0]$$

$$XX^T = \begin{bmatrix} a \\ 0 \end{bmatrix} [a \quad 0] = \begin{bmatrix} a^2 & 0 \\ 0 & 0 \end{bmatrix} = A$$

eigenvalue $|A - \lambda I| = 0$

$$\begin{vmatrix} a^2 - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$(a^2 - \lambda)(-\lambda) = 0$$

$\lambda = 0$, $\lambda = a^2$ (one non-zero)

for $n = 3$ $(n > 1)$

$$X = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \qquad X^T = [a \quad 0 \quad 0]$$

$$XX^T = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigen value

$$\begin{vmatrix} a^2 - \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$\lambda = 0, 0, a^2$ (one non-zero)

41. J^P for the ground state of the $^{13}C_6$ nucleus is :

(a) 1^+ (b) $\frac{3^-}{2}$

(c) $\frac{3^+}{2}$ (d) $\frac{1^-}{2}$

$^{13}C_6 \Rightarrow P = 6$, $N = 7$

$1S_{1/2}^2 \; 1P_{3/2}^4 \; 1P_{1/2}^1$

$\Rightarrow J = \frac{1}{2}$

$P = (-1)^l = (-1)^1 = -1$

$\Rightarrow J^P = \frac{1}{2}^-$

42. A uniform solid cylinder is released on a horizontal surface with speed 5 m/s without any rotation (slipping without rolling). The cylinder eventually starts rolling without slipping. If the mass and radius of the cylinder are 10 gm and 1 cm respectively, the final linear velocity of the cylinder is ………… m/s. (up to two decimal places).

Angular momentum

L = angular momentum about CM + $I\omega$

$$L = L_{CM} + I\omega$$

$$mvr = mV_{cm}r + \left(\frac{1}{2}mr^2\right)\omega$$

$$= mV_{cm}r + \frac{1}{2}mr^2\frac{V_{cm}}{r}$$

$$= mV_{cm}r + \frac{1}{2}mV_{cm}r$$

$$mvr = \frac{3}{2}mV_{cm}r$$

$$V_{cm} = \frac{2}{3}V$$

$$V_{cm} = \frac{2}{3}\times 5$$

$$V_{cm} = 3.33\ m/sec$$

43. The energy density and pressure of a photon gas are given by $u = aT^4$ and $P = u/3$, where T is the temperature and a is the radiation constant. The entropy per unit volume is given by $\alpha a T^3$. The value of α is ……………. . (up to two decimal places).

$$u = aT^4 \quad , \quad P = \frac{u}{3}$$

$$S = \alpha a T^3$$

$$du = +TdS - PdV$$

$$T = \left(\frac{\partial U}{\partial S}\right)_V$$

$$dS = \frac{du}{T}$$

$$d(\alpha a T^3) = \frac{d(aT^4)}{T}$$

$$3\alpha a T^2 = \frac{4aT^3}{T}$$

$$3\alpha a T^2 = 4aT^2$$

$$\alpha = \frac{4}{3} = 1.33$$

44. Which one of the following gases of diatomic molecules is Raman, infrared, and NMR active?

(a) ${}^{1}H-{}^{1}H$ (b) ${}^{12}C-{}^{16}O$

(c) ${}^{1}H-{}^{35}C1$ (d) ${}^{16}O-{}^{16}O$

ANS-(C)

45. The π^+ decays at rest to μ^+ and ν_μ. Assuing the neutrino to be massless, the momentum of the neutrino is .... MeV/c. (up to two decimal places) $(m_\pi = 139\ MeV/c^2, m_\mu = 105 MeV/c^2)$

$$E_\nu = \frac{(m_\pi^2 + m_\nu^2 - m_\mu^2)c^2}{2m_\pi}$$

$$= \frac{(m_\pi + m_\mu)(m_\pi - m_\mu)}{2m_\pi} c^2 \quad [\because m_\nu = 0]$$

$$= \frac{(139 + 105)(139 - 105)}{2 \times 139} \text{ MeV}$$

$$= \frac{244 \times 34}{2 \times 139} \text{ MeV}$$

$$E^2 = p^2c^2 + m_0^2c^4$$

$$E_\nu^2 = p_\nu^2 c^2$$

$$p_\nu = \frac{E_\nu}{c}$$

$$= \frac{244 \times 34}{2 \times 139} \frac{\text{MeV}}{c}$$

$$= 29.84 \frac{\text{MeV}}{c}$$

46. Using Hund's rule, the total angular momentum quantum number J for the electronic ground state of the nitrogen atom is :

(a) 1/2 (b) 3/2

(c) 0 (d) 1

$_7N \Rightarrow 1s^2 2s^2 2p^3$

p-subshell is half filled

$\Rightarrow {}^4S_{3/2}$

$J = \frac{3}{2}$

47. Which one of the following operators is Hermitian?

(a) $i\dfrac{(p_x x^2 - x^2 p_x)}{2}$ (b) $i\dfrac{(p_x x^2 + x^2 p_x)}{2}$

(c) $e^{i p_x a}$ (d) $e^{-i p_x a}$

(a) $\dfrac{i(P_x x^2 - x^2 P_x)}{2} = \hat{A}$

$\hat{A}^{+} = \left[\dfrac{i(P_x x^2 - x^2 P_x)}{2}\right]^{+}$

$= -\dfrac{i}{2}\left[(x^{+})^2(P_x^{+}) - (P_x^{+})(x^{+})^2\right]$

$= -\dfrac{i}{2}\left[x^2 P_x - P_x x^2\right] \qquad \left[\because \begin{matrix} x^{+} = x \\ P_x^{+} = P_x \end{matrix}\right]$

$\hat{A}^{+} = \dfrac{i}{2}\left[P_x x^2 - x^2 P_x\right]$

$\hat{A}^{+} = \hat{A}$ option (a) is correct

48. The real space primitive lattice vectors are $\vec{a}_1 = a\hat{x}$ and $\vec{a}_2 = \dfrac{a}{2}(\hat{x} + \sqrt{3}\hat{y})$. The reciprocal space unit vectors $\vec{b}_1$ and $\vec{b}_2$ for this lattice are, respectively

(a) $\dfrac{2\pi}{a}(\hat{x} - \dfrac{\hat{y}}{\sqrt{3}})$ and $\dfrac{4\pi}{a\sqrt{3}}\hat{y}$

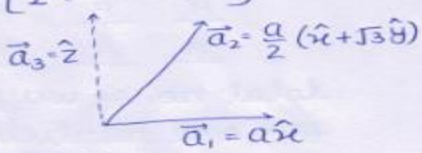
(b) $\dfrac{2\pi}{a}(\hat{x} + \dfrac{\hat{y}}{\sqrt{3}})$ and $\dfrac{4\pi}{a\sqrt{3}}\hat{y}$

(c) $\dfrac{2\pi}{a\sqrt{3}}\hat{x}$ and $\dfrac{4\pi}{a}(\dfrac{\hat{x}}{\sqrt{3}} + \hat{y})$

(d) $\dfrac{2\pi}{a\sqrt{3}}\hat{x}$ and $\dfrac{4\pi}{a}(\dfrac{\hat{x}}{\sqrt{3}} - \hat{y})$

Sol :-

$\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = a\hat{x} \cdot \left[\dfrac{a}{2}(\hat{x} + \sqrt{3}\hat{y}) \times \hat{z}\right]$



$\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = a\hat{x} \cdot \dfrac{a}{2}\left[-\hat{y} + \sqrt{3}\hat{x}\right]$

$= \dfrac{\sqrt{3}}{2} a^2$

Now $\vec{b}_1 = \dfrac{2\pi(\vec{a}_2 \times \vec{a}_3)}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$

$= \dfrac{2\pi \times \frac{a}{2}(-\hat{y} + \sqrt{3}\hat{x})}{\sqrt{3}/_2\, a^2}$

$= \dfrac{2\pi}{\sqrt{3}a}(\sqrt{3}\hat{x} - \hat{y})$

$\vec{b}_1 = \dfrac{2\pi}{a}(\hat{x} - \dfrac{1}{\sqrt{3}}\hat{y})$

$\vec{b}_2 = \dfrac{2\pi(\vec{a}_3 \times \vec{a}_1)}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$

$= \dfrac{2\pi(\hat{z} \times a\hat{x})}{\frac{\sqrt{3}}{2}a^2}$

$= \dfrac{4\pi a\hat{y}}{\sqrt{3}a^2}$

$\vec{b}_2 = \dfrac{4\pi}{\sqrt{3}a}\hat{y}$

option – (a)

49. Consider two particles and two non-degenerate quantum levels 1 and 2. Level 1 always contains a particle. Hence, what is the probability that level 2 also contains a particle for each of the two cases :

(i) when the two particles are distinguishable and (ii) when the two particles are bosons?

(a) (i) 1/2 and (ii) 1/3

(b) (i) 1/2 and (ii) 1/2

(c) (i) 2/3 and (ii) 1/2

(d) (i) 1 and (ii) 0

Sol :-

for distinguishible

A B

B A AB

total no. of ways = 3

for one particle in level 2 = 2

so, probability = $\frac{2}{3}$

for Bosons (indistinguishable)

probability = $\frac{1}{2}$

50. A person weights w_p at Earth's north pole and w_e *at the* equator. Treating the Earth as a perfect sphere of radius 6400 km, the value $100 \times (w_p - w_e)\, w_p$ is ....(up to two decimal places). (Take $g = 10\ ms^{-2}$).

Sol :- Effect of rotation of earth on gravity

$$g' = g - \omega^2 R \cos\lambda$$

at pole $\lambda = 90^\circ$, $g' = g = g_p$

at equator $\lambda = 0$, $g' = g - \omega^2 R = g_e$

$$\frac{g_p - g_e}{g_p} = \frac{g - (g - \omega^2 R)}{g} = \frac{\omega^2 R}{g}$$

So, $$100 \times \left(\frac{\omega_p - \omega_e}{\omega_p}\right) = 100 \times \frac{\omega^2 R}{g}$$

$R = 6400\,km$ $\quad g = 10\,m/s^2$ $\quad \omega = 7.26 \times 10^{-5} \frac{rad}{sec}$

$$100 \times \left(\frac{\omega_p - \omega_e}{\omega_p}\right) = \frac{\omega^2 R}{g} \times 100$$

$$= \frac{(7.26 \times 10^{-5})^2 (6400\,km) \frac{rad}{sec}}{10\,m/s^2}$$

$$= 0.338$$

51. The geometric cross-section of two colliding protons at large energies is very well estimated by the product of the effective sizes of each particle. This is closest to

(a) 10 b (b) 10 mb

(c) 10 μb (d) 10 pb

ANS-(B)

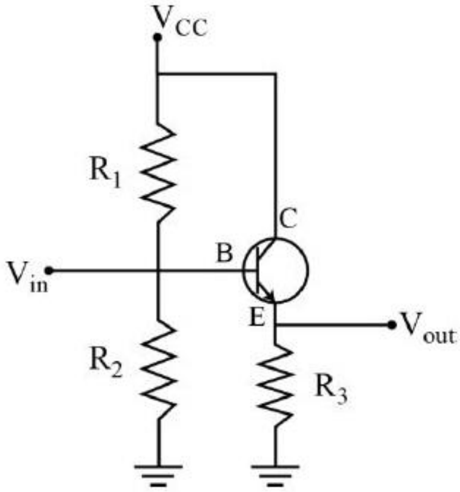
52. For the transistor amplifier circuit shown below with $R_1 = 10k\Omega$, $R_2 = 10k\Omega$, $R_3 = 1k\Omega$, and $\beta = 99$. Neglecting the emitter diode resistance, the input impedance of the amplifier looking into the base for small ac signal is ………….. $k\Omega$. (up to two decimal places).

According to Voltage divider biasing

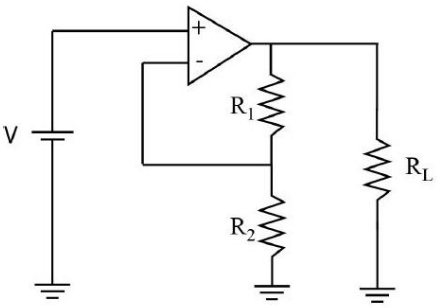
input impedence

$$R_{th} = \frac{R_1 \times R_2}{R_1 + R_2}$$

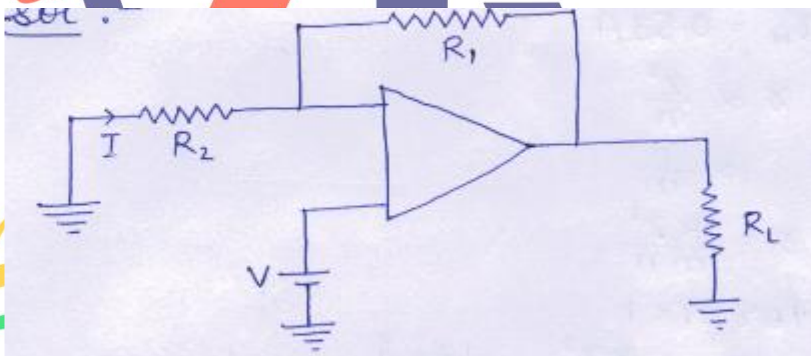
$$= \frac{10 \times 10}{10 + 10} = \frac{100}{20} = 5\,k\Omega$$



53. Consider an ideal operational amplifier as shown in the figure below with $R_1 = 5k\Omega,\ R_2 = 1k\Omega,\ R_L = 100k\Omega$. for an applied intput voltage V = 10 mV, the current passing through R_2 is .... μA. (up to two decimal places).



Sol.



$$I = \frac{0 - V}{R_2} = \frac{-V}{R_2}$$

$$I = \frac{-10\,mV}{1 \times 10^3}$$

$$I = 10\,\mu A$$

54. Consider the differential equation $dy/dx + y\tan(x) = \cos(x)$. If $y(0) = 0,\ y(\pi/3)$ is .... (up to two decimal places).

Sol^n:-

$$\frac{dy}{dx} + y\tan x = \cos x$$
$$\Rightarrow P \qquad \Rightarrow Q$$

$$I.F. = e^{\int P dx}$$
$$= e^{\int \tan x dx} = e^{\log(\sec x)}$$
$$I.F. = \sec x$$

Now

$$y \times I.F. = \int Q \times I.F.\, dx + C$$
$$y \times \sec x = \int \cos x \times \sec x\, dx + C$$
$$y \times \sec x = x + C$$

at $x = 0$, $y = 0$

$$0 \times \sec 0 = 0 + C$$
$$C = 0$$
$$\Longrightarrow y \sec x = x$$
$$y(\pi/3) = \frac{\pi/3}{\sec(\frac{\pi}{3})}$$
$$y(\tfrac{\pi}{3}) = \frac{2\pi}{3\times 2} = 0.52$$

55. Positronium is an atom made of an electron and a positron. Given the Bohr radius for the ground state of the Hydrogen atom to be 0.53 Angstroms, the Bohr radius for the ground state of positronium is …………… Angstroms. (up to two decimal places).

$$r_H = 0.53\,\text{Å}$$
$$r \propto \frac{Z^2}{n}$$
$$\propto \frac{1}{m}$$
$$r = \frac{kZ^2}{mn}$$

for $n = 1$

$$r_H = \frac{kZ^2}{m_e} = 0.53\,\text{Å}$$

for positronium

$$m = \frac{m_e \times m_e}{m_e + m_e} = \frac{m_e}{2}$$

for H-atom

$$m = \frac{m_p \times m_e}{m_p + m_e} = m_e$$

$$r_{positronium} = \frac{k}{m_{pos}} \frac{Z^2}{n}$$
$$= \frac{kZ^2}{m_e/2} \qquad [\because n = 1]$$
$$= \frac{2kZ^2}{m_e}$$
$$= 2 \times 0.53\,\text{Å}$$
$$= 1.06\,\text{Å}$$

56. The ninth and the tenth of this month are Monday and Tuesday ………… .

(a) figuratively (b) retrospectively

(c) respectively (d) rightfully

ANS-(C)

57. It is .... to read this year's text book .... the last year's ....

(a) easier, than (b) most easy, than

(c) easier, from (d) easiest, from

ANS-(A)

58. A rule states that in order to drink beer, one must be over 18 years old. In a bar, there are 4 people. P is 16 years old. Q is 25 years old, R is drinking milkshake and S is drinking a beer. What must be checked to ensure that the rule is being followed?

(a) Only P's drink (b) Only P's drink and S's age

(c) Only S's age (d) Only P's drink, Q's drink and S's age

ANS-(B)

59. Fatima starts from Point P, goes North for 3 km, and then East for 4 km to reach point Q. She then turns to face point P and goes 15 km inthat direction. She then goes North for 6 km. How far is she from point P, and in which direction should she go to reach point P?

(a) 8 km, East (b) 12 km, North

(c) 6 km, East (d) 10 km, North

ANS-(A)

60. 500 students are taking one or more courses out of chemistry, Physics, and Mathematics. Registration records indicate course enrolment as follows : Chemistry(329), Physics (186), Mathematics (295), Chemistry an Physics (83), Chemistry and mathematics (217), and Physics and Mathematics (63). How many students are taking all 3 subjects?

(a) 37 (b) 43

(c) 47 (d) 53

ANS-(D)

61. "If you are looking for a history of India, or for an account of the rise and fall of the British Raj. or for the reason of the cleaving of the subcontinent into two mutually antagonistic parts and the effects this mutilation will ave in the respective sections, and ultimately on Asia, you will not find it in these pages; for though I have spent a lifetime in the country. I lived too near the seat of events, and was too intimately associated with the actors, to get the perspective needed for the impartial recording of these matters".

Which of the following statements best reflects the author's opinon?

(a) An intimate association does not allow for the necessary perspective.

(b) Matters are recorded with an impartial perspective.

(c) An intimate association offers an impartial perspective.

(d) Actors are typically associated with the impartial recording of matters.

ANS-(A)

62. Each of P, Q, R, S, W, X, Y and Z has been married at most once. X and Y are married and have two children P and Q. Z is the grandfather of the daughter S of P. Further. Z and W are married and are parents of R. Which one of the following must necessarily be FALSE?

(a) X is the mother-in-law of R

(b) P and R are not married to each other

(c) P is a son of X and Y

(d) Q cannot be married to R

ANS-(D)

63. 1200 men and 500 women can build a bridge in 2 weeks, 900 men and 250 women will take 3 weeks to build the same bridge. How many men will be needed to build the bridge in one week?

(a) 3000 (b) 3300

(c) 3600 (d) 3900

ANS-(C)

64. The number of 3-digit numbers such that the digit 1 is never to the immediate right of 2 is

(a) 781 (b) 791

(c) 881 (d) 891

ANS-(C)

65. A contour line joins locations having the same height above the mean sea level. The following is a contour plot of a geographical region. Contour lines are shown at 25 m intervals in this plot.

Which of the following is the steepest path leaving from P?

(a) P to Q (b) P to R

(c) P to S (d) P to T

ANS-(B)